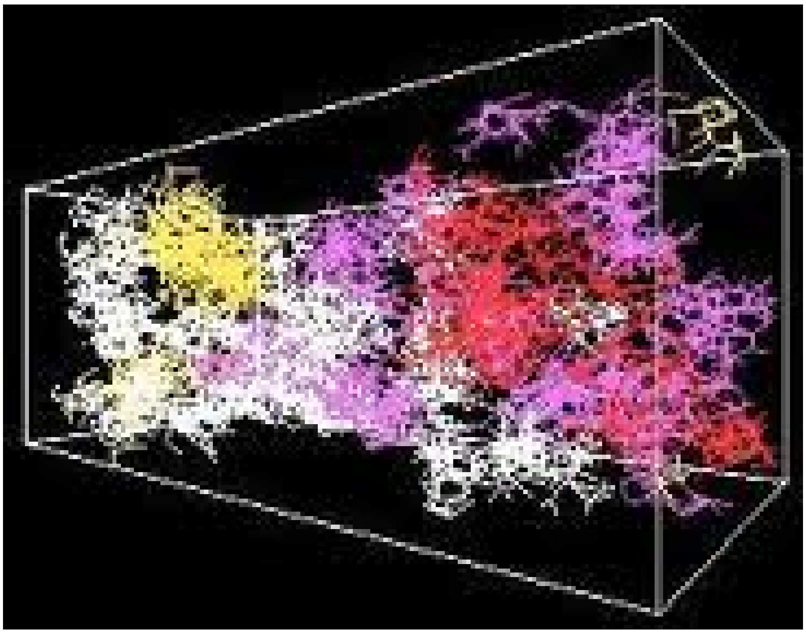

[Anticipiamo l’introduzione di Paolo Maria Mariano al numero tematico di ottobre 2012 della rivista scientifica dell’editore statunitense Wiley, “Mathematical Methods in Applied Sciences”, dedicato a “Perspectives in Continuum Mechanics”.]

With the words *continuum mechanics* we indicate a classical field theoretic representation (long wavelength approximation) of the macroscopic mechanical behavior of bodies extended in space, under the action of external agencies. In this setting the description of changes in placements including strain effects plays a prominent rôle.

Bodies populating the physical world are the resultant of intricate architectures of entangled molecules or regular atomic arrays. And phenomena at various spatial and/or temporal scales concur in determining their macroscopic mechanical behavior. In constructing relevant interpretative models we make choices by selecting prominent aspects in the set of phenomena that we want to describe. In doing these choices we are addressed by empirical data but, in turn, design and development of experiments are driven by pre-existing theoretical views on the phenomenon they are referred to.

The traditional format of continuum mechanics of bodies that can suffer strain, as axiomatized in a neoclassical solid structure by the extended foundational work of Truesdellian school, has a minimalist view on the problem. It follows three steps in a logical hierarchy: (1) The morphology of a body is described by the sole regularly open region in space that it can occupy at least in principle. So the description of low scale geometries is neglected. Geometric points represent pieces of matter, each one imagined as a black box -- call them material elements. (2) Actions are distinguished in bulk and contact classes. The first class refers to gravitational, inertial, electromagnetic ones. The second class includes actions generated by the contact with the environment and between neighboring parts of a body. In the latter case we speak of inner actions due to crowding and shearing of material elements. Balances of bulk and contact actions are determined by the invariance under isometric changes of spatial frames of the external power that they perform, a power defining them. Integral balances then correspond to Killing fields of the metric in the physical space, which is chosen Euclidean. (3) Possible constitutive structures (state functions) are restricted a priori by the second law of thermodynamics, even in its isothermal version of mechanical dissipation inequality. Such a law can be used also to investigate stability questions. Moreover it is exploited as a criterion of admissibility for shock waves.

A reader just glimpsing through previous items could think that the development of a model for some class of deforming bodies could emerge just by the appropriate choice of constitutive structures. Although in prominent cases the reasoning applies, in general this point of view is at least ingenuous with respect to perspectives offered by the offsprings of condensed matter physics. Large classes of natural bodies and the behavior of materials constructed anew in chemical industry to satisfy specific technological demands display non-trivial interplay among phenomena at various scales (as recalled at the beginning of these notes) that can be hardly portrayed by using the traditional scheme of continuum mechanics, even with efforts of phantasy in selecting intricate constitutive structures.

Examples are repeatedly presented in literature. They refer to the arrangements of stick molecules with end-to-tail symmetry in liquid crystals, the polarization in ferroelectrics and the corresponding magnetization in magnetostrictive materials, the atomic flips in quasicrystals, the creation and annihilation of molecular entanglements in polymeric materials etc. For them, multi-field and multi-scale representations of condensed matter appear necessary. Changes in the standard paradigm sketched above start then from the first item: the description of the body morphology. Instead of being considered as a black box, the generic material element can be, more realistically, viewed as a system at an (exploded) appropriate sub-scale in space. As a consequence, the submanifold of the ambient space, selected commonly as reference macroscopic place for the body, has to be considered (together with the time interval), as the base of a fiber bundle with typical fiber the direct product of the physical ambient space (an Euclidean point space in the classical setting, a Riemannian manifold, more in general) and a finite-dimensional, differentiable, complete manifold, the elements of which are descriptors of peculiar geometric features of the low-scale material architecture -- once we presume for simplicity a statistical homogeneity in the typology of material structure. Configurations are then sections of this fiber bundle. At each pair place-instant on the basis, we are then able to assign the actual place and the corresponding value of the descriptor of the inner material morphology. The representation becomes then multi-field and multi-scale.

Over this fiber bundle, actions are then appropriately represented by elements of the pertinent cotangent bundle for they are defined by the power they develop in the rates at which macroscopic and microscopic material morphologies change. For the actions related with the low-scale events (we call them microstructural actions just to give a name evoking their nature) we can, at the end, adopt the same traditional classification into bulk and contact families. To find appropriate balances for both families of standard and microstructural actions, without postulating them, we can follow a procedure based on the invariance of the external power of all actions. But to develop it, we need now an extended notion of observer considered as a representation of the whole fiber bundle where changes in the representation of one factor of the fiber (the ambient physical space) have influence in a precise sense on companion changes on the manifold of microstructural shapes. The determination of constitutive structures is based not only on the use of a version of the second law of thermodynamics including microstructural actions, but also on appropriate discrete-to-continuum descriptions of the low-dimensional material morphology. In this setting, a non-trivial question is the expression of the structure of the microstructural actions (a Cauchy theorem for the representation of contact microstructural interactions in the abstract setting seems not yet available).

The framework furnishes also synthetic schemes describing bodies with one or two very low dimensions like thin films, nanotubes, nanorods. It can be also a basis to interpret multi-scale kinetic views on the dynamics of granular matter.

But the story does not end here. A not explicitly declared assumption in the standard format and the extended one sketched above is that the reference place is fixed once and for all. When defects occur -- they are structural alterations of a material structure that we decide to be defect free, and in this sense to be defective is a relative concept -- and evolve, or a body undergoes growth and remodeling, a picture of these events in the reference place seems to be appropriate in order to distinguish them from the standard deformation. So, it is natural to think of a family of possible reference places for a body, parametrizing it by measures in appropriate cases or by time. When we adopt the last choice, a notion of relative power is appropriate to derive from a unique source, by imposing its invariance under changes in observers intended as above, both the balances of standard and microstructural actions and the ones associated with mechanisms pictured in the reference place.

Another issue deals with general proofs of covariance of the model-building framework described above. In this non-relativistic setting, we can speak about covariance in appropriate circumstances dealing with the derivation of balance equations, the restriction of possible constitutive structures, the ambient space where we evaluate current places. And not always all these aspects appear together. As regards balance equations, covariance is invoked when we prove that pointwise balances can emerge from some invariance requirement of a certain basic principle (it can be the balance of energy or the second law of thermodynamics in appropriate fashion) with respect to changes in observers governed by the action of the group of diffeomorphisms of the fiber bundle mentioned above, instead of resorting just to the action of isometries in the fiber factor describing the ambient space, with related changes in the other factor. The generic changes in observers just mentioned can be used to restrict possible constitutive structures and are essential when we consider the ambient physical space (the one where we count just macroscopic places) as a generic Riemannian manifold instead of considering just the n-dimensional Euclidean space. Covariance issues pave the way to relativistic versions of continuum mechanics, although in that case the view should be perhaps reversed because we should refer to back-to-label maps associating world lines to material points since we should not be able to distinguish single places and instants.

In the traditional format of continuum mechanics, and also in the extended framework taking into account the complexity of the inner material morphology, the independence of the derivation of balance equations from constitutive issues is stressed. It is strictly related to the rigid structure of the n-dimensional Euclidean physical space, irrespective of the nature of the finite-dimensional differentiable manifold where we collect descriptors of the inner material morphology. When we involve covariance, such a distinction seems to be lost because, in invoking the first principle of thermodynamics or the second one, the procedures we have at hands actually require the specification of the state variables entering the internal energy (in the first principle) or the free energy (in the second one).

The list of possible fields of investigation must not forget thermodynamics, even starting from basic issues related to a definition of temperature in bodies with complex material structure, based on microscopic evaluations of the physical mechanisms, or the contribution of low-scale events to entropy fluxes and sources. Incidentally, even if we just call upon a phenomenological concept of temperature, we can find circumstances where the occurrence of microstructural events can naturally determine finite speed heat propagation, in contrast with standard Fourier's approach.

In the whole settings sketched so far, perspectives on the wide landscape of mechanics are manifold. Chapters of (above all) differential geometry and functional analysis are necessary tools to trace theoretical research paths. Along the way, we should take into account two remarks. (1) The development of the general structures of continuum mechanics and the construction of models for specific classes of materials do not reduce simply to a trite manipulation or juxtaposition of pieces of existing models without stringent physical necessity. (2) Models in mechanics are not just a pretext to justify more or less difficult exercises in analysis or differential geometry. Rather the analyses should be always strictly linked to the physical significance of the objects used. And it should be a must.

Models are, in fact, representations of the empirical world. We find convenient to express them in mathematical terms because this way we use a language which allows us to express qualitative and quantitative statements by resorting just to its own structures. A problem is then the choice between two competing models with similar results or among models with slight differences. Heinrich Hertz, in his 1894 book *Die Prinzipien der Mechanik*, supplied by a the subtitle *In neuen Zusammenhange dargestell*, suggested as criteria of empirical adequacy permissibility, correctness, and appropriateness. Permissibility is intended as non-contradiction with the laws of thought, a neo-Kantian point of view relying on the assumption of a pre-existing non-problematic logical structure of the way we think. Then correctness is a requirement of physical necessity. Finally, appropriateness is intended as the capability to capture the largest number of features that we consider essential with the simplest logical way (which is not strictly related with the simplicity of the mathematical tools adopted). These last aspects introduce an aesthetic factor connected with the way we intend simplicity which translates in elegance of theoretical and related computational constructs.

In building up a mechanical model, it is hard to think of avoiding to take into account Hertz's suggestions, which have been clear precursors of contemporary interpretations proposed in philosophy of science on the nature of physical theories. But the criteria he has proposed do not have and cannot probably have an unquestionable formal definition like the one of a mathematical object. So they have to be intended, together with the analyses produced lately in the same topic, as hints addressing us along the way. The judgement of the quality of mechanical models is then a question of sensibility and culture of the scholar tackling the problem. And it is just sensibility, supported by culture, the guide opening perspectives in the tentative that we do in trying to investigate the nature of the physical world around us, being conscious that the enterprise is without end.

Works going along paths where we can see perspectives in continuum mechanics are collected in this issue, instead of being scattered along regular issues of the journal. They are a right mix of physical views and mathematical techniques, each one with its proper attention to the foundations of the subfield they are included in. Such a kind of attention should be always present in mathematical methods in the applied sciences for it is an attitude allowing us to develop models able to represent adequately even intricate circumstances (think of the increasing demand of sophisticated performances by materials and structures used in industry).

Here below I furnish a fugitive overview on the set of contributions to this issue. The order is the one of appearance in the table of contents.

In describing properties of crystalline materials, Rachel Nicks and Gareth P. Parry (*On symmetries of crystals with defects related to a class of solvable groups (S[illegible])*, this issue) focus attention on atomic lattices. For them, we can imagine often that every crystalline cell (the material element, collapsed in the continuum modeling onto a point) deforms homogeneously. This is the so-called Cauchy-Born rule. By using it we can justify, on the basis of an atomistic model, the dependence of the macroscopic elastic energy density on the right Cauchy-Green tensor, i.e. the pull-back on reference place of the spatial metric along a transplacement from that place to the current one. The rule, however, does not always hold when we have slips of atomic planes. Nicks and Parry take into account the case and show that at macroscopic scale we need to consider, in presence of slips, the elastic energy density as a function of point values of appropriate vector fields and the dislocation density tensor. This way they generalize in a sense the standard continuum description of (elastic) crystals.

At a scale larger than the one of atomic lattices, but also smaller with respect to the macroscopic deformation, phenomena influencing drastically the macroscopic mechanical behavior may occur. Thomas J. Pence (*On the formulation of boundary value problems with the incompressible constituents constraint in finite deformation poroelasticity*, this issue) considers the special case of elastic materials with voids filled by an incompressible fluid and proposes a generalization of the so-called saturation stress. Such a stress is independent of the pressure associated with the internal constraint that the constituents are individually incompressible and that mixing occurs without the formation of voids. And it is a basic tool in discussing boundary conditions for the whole mixture.

Porosity is just one possible type of material microstructure. And it can be represented not only by taking as a descriptor of the material morphology the void volume fraction, but also the interface between the pore and the material around it. Such an approach is particularly useful in presence of phase transitions when we want to analyze the evolution of drops inside a surrounding medium. In two dimensional ambient space, Daniel Ševčovic and Shigetoshi Yazaki (*Computational and qualitative aspects of motion of plane curves with a curvature adjusted tangential velocity*, this issue) offer analyses constituting a descriptive and predictive tool when these interfaces are closed Jordan curves endowing normal motion. For them they analyze first the case in which the normal velocity depends on curvature modulus and curvature average. And they prove, locally in time, existence and uniqueness of classical smooth solutions. As a constitutive assumption, they also presume that the normal velocity is determined by two mechanisms: a local one which is function of the placement, the curvature, and the tangent angle, and a non-local one depending on the curve total length, the enclosed area, the bending elastic energy of the curve -- it does not sustain shear by assumption. So they consider various cases, paying attention to the analysis of gradient flow for the isoperimetric ratio.

Micro-interfaces can coalesce up to the formation of macroscopic discontinuity surfaces which can be coherent or incoherent. In the first case just folding across them is permitted. The second case includes relative slips between the two pieces of material concurring to define the surface itself. Examples are manifold. In particular, incoherent interfaces can occur in elastic-plastic materials, due to slip and misorientation of grains, which are crystalline aggregates in case of metallic alloys. Anurag Gupta and David J. Steigmann (*Plastic flow in solids with interfaces*, this issue) propose a framework able to describe these kind of interfaces when they suffer diffusionless normal motion in isothermal conditions, taking into account their incoherency and the presence of bulk plastic flows in the surrounding material.

The elastic-plastic behavior of condensed matter, especially in presence of large strains, has not a unique representative format, even in the case of simple elastic materials undergoing an elastic-to-plastic phase transition toward perfect plasticity. A possible non-standard description of plastic phenomena in solids is presented by Tamás Fülöp and Péter Ván (*Kinematic quantities of finite elastic and plastic deformation*, this issue). They act directly on a space-time structure and consider transplacements as orientation preserving smooth maps between a three-dimensional simply-connected complete smooth manifold and another smooth manifold. They prescribe a Ricci-flat metric on the material manifold (the way to assign it is clear in case of crystal structures, for it is generated by the local crystal optical axes, but could be problematic -- in the sense its choice could be not unique -- for amorphous materials). So they develop analyses describing the occurrence of plastic flows by time-dependence of the material metric (this last aspect agrees with a 1998 C. Miehe's proposal with differences determined by the settings). Torsion in the structure of material manifold could be considered in case of bodies with continuous distributions of dislocations.

Intricate problems may occur even in the traditional setting of linear elasticity when we imagine to design anew a material to be able to reach certain performances. It is the case of composites that we want to construct along some prescription of optimality in the geometry of their inner (reinforcing) structure, above all when such materials admit a threshold across which there is a transition toward (say) plastic or brittle behavior. Anita Catapano, Boris Desmorat, and Paolo Vannucci (*Invariant formulation of phenomenological failure criteria for orthotropic sheets and optimisation of their strength*, this issue) approach the strength optimization of linear elastic plane orthotropic bodies, admitting polynomial-type failure criteria, by expressing the criteria themselves in terms of invariants. The procedure allows them to reduce the variables to be optimized just to the orthotropy orientation and indicate a point of view which can be utilized for other types of material anisotropy.

Thresholds to linear elastic material states do not give information on post-critical behavior. Standard linear elasticity has to be enriched for the purpose. For example, in case of elastic-brittle post-critical behavior, besides the way to describe the nucleation of a crack (there are alternatives, of course, even with variational nature), we have simply to add, for example, conditions avoiding the interpenetration of the two margins of the crack. So, in addition to modeling issues, a number of analytical problems emerges. In case of linear elastic materials with cracks, Dorothee Knees and Adreas Schröder (*Global spatial regularity for elasticity models with cracks, contact and other nonsmooth constraints*
, this issue) show conditions for global higher differentiability in Besov spaces of the displacement field. In addition, they discuss regularity issues for other cases like contact of a body with a nonsmooth rigid foundation, the behavior of materials with Tresca-type friction, the occurrence of shape-memory (dissipative but reversible) effects in metallic alloys.

Materials may show dependence of the state functions on the past history of deformation and/or other state variables. The standard description of viscoelasticity is a paradigmatic example. An efficient formal tool to express non-locality in time (as memory is, in fact) is fractional calculus. Besides viscosity, such a tool can also be exploited in the description of convection-diffusion phenomena, like for example the propagation of pollulants in atmosphere. For the convection diffusion equation, having a view toward the construction of useful numerical schemes, Jacky Cresson, Isabelle Greff, and Pierre Inizan (*Lagrangian for the convection--diffusion equation*, this issue) suggest that their solution can be considered as the critical points of a Lagrangian expressed in terms of fractional derivatives. The essential tool is a decomposition time by time of the state space into accessible and non-accessible parts (past and future, indeed) which allow them to take into account irreversibility.

Finally, almost entirely from the analytical side, Richard Marchand, Tim McDewitt, and Roberto Triggiani (*An abstract semigroup approach to the third-order Moore--Gibson--Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability*, this issue) start from the Moore-Gibson-Thompson equation, one of the possible descriptions of non-linear acoustic phenomena, based on the acceptance of Maxwell-Cattaneo rule for heat propagation, and analyze an abstract generalization of it: a third-order equation on a (separable) Hilbert space, involving a non-negative self-adjoint operator with compact resolvent. A number of properties emerges from operator-theoretic semigroup based analyses, visualized in one-dimensional numerical examples.

All these papers show possible views on aspects of continuum mechanics and can be source of future work. They also testify a tendency of this journal to include more and more works paying non-trivial attention to the foundations of the areas they are referred to, with the consciousness that concrete and effective developments in applied sciences rely on a deep comprehension and command of the inner nature of the models involved and the techniques utilized.